数理解析研究所講究録
292巻 1977年 90-97

ON EXTENDING FUNCTIONALS ON COMMUTATIVE SEMIGROUPS

Yuji Kobayashi

In the present report we will outline some of the studies on homomorphisms of commutative semigroups into the additive group of real numbers; when they are extensible or when they exist. Most of the proofs are omitted and the reader should refer to the literature cited.

1. Introduction. Let G be a commutative semigroup. A homomorphism of G into the additive group $\mathbb{R}$ of all real numbers is called a functional of G. Let H be a subsemigroup of G. When is a functional of H extended to a functional of G? The answer is easy. In fact, it is almost always (for example, always if G is cancellative) extensible as the following proposition shows. The reason is based on the fact that $\mathbb{R}$ is a divissible abelian group.

Proposition 1. Let G be a commutative semigroup and H its subsemigroup. Let f be a functional of H. Then f is extensible to G if and only if f satisfies

(1) $$h_1g=h_2g,\ h_1,h_2\in H,\ g\in G \Longrightarrow f(h_1)=f(h_2).$$

Our problem next is extending some types of functionals under some suitable conditions. We consider two cases: (i) functionals bounded by some functions from the both sides, (this case brings us some Hahn-Banach type theorems,) (ii) non-negative functionals (non-negative real valued functionals).

2. Hahn-Banach type theorem. There are many versions and extended forms of the Hahn-Banach theorem on linear spaces. Kaufman [2], [3] gave some Hahn-Banach type extension theorem on commutative semigroups. Fuchssteiner [1] established it as an elegant theorem (Sandwich theorem) and deduced many related results from it. Here we will give a new theorem from which their results are deduced (see [6] for details).

Let θ and f be functions of a commutative semigroup G into $\mathbb{R}$. f is called a lower (resp. upper) θ-function of G if for all $x,y \in G$

(2) $$f(x)+f(y) \leq f(xy) \leq f(x)+\theta(y)$$

$$(\text{resp. } f(x)+f(y) \geq f(xy) \geq f(x)+\theta(y)).$$

A function f of G is called homogeneous if for any $x \in G$ and $n \in \mathbb{Z}_+$(the set of all positive integers)

(3) $$f(x^n) = nf(x).$$

We can always homogenize a lower (upper) θ-function by the following lemma.

Lemma 1. Let f be a lower (resp. upper) θ-function of G. Then there exists the limit

(4) $$\varphi(x) = \lim_{n\to\infty} \frac{f(x^n)}{n},$$

for every $x \in G$, and φ is a homogeneous lower (resp. upper) θ-function satisfying $f \leq \varphi$ (resp. $f \geq \varphi$).

Theorem 1. Let G be a commutative semigroup and H its subsemigroup. Let f be a lower (resp. upper) θ-function of G such that $f|_H$ is a functional of H. Then there exists a functional $\bar{f}$ of G such that $f \leq \bar{f} \leq \theta$ (resp. $\theta \leq \bar{f} \leq f$) and $f|_H = \bar{f}|_H$.

We give a sketch of the proof. Let (g, K) be a couple of a lower θ-function f and a subsemigroup K such that $g \geq f$, $K \supset H$, $g|_H = f|_H$ and $g|_K$ is a functional of K. Let $(\bar{f}, \bar{K})$ be a maximal element (the existence is assured by Zorn's lemma) in the couples in the sense of the order:

$$(5) \qquad (g, K) \geq (g', K') \Longleftrightarrow g \leq g',\ K \subset K' \text{ and } g|_K = g'|_K.$$

If $\bar{H} \neq G$, we can find $x_o \in G \setminus \bar{H}$, $h_o \in \bar{H}$ and $n_o \in \mathbb{Z}_+$ such that

$$(6) \qquad \bar{f}(x_o^{n_o} h_o) > n_o \bar{f}(x_o) + \bar{f}(h_o).$$

$f_n(x) = \bar{f}(x h_o^n) - n\bar{f}(h_o)$ is monotone increasing on n for every $x \in G$ and bounded by $\theta(x)$. Hence there exists the limit $g(x) = \lim_{n \to \infty} f_n(x)$. Then it is proved that g is a lower θ-function such that $g|_{\bar{H}}$ is a functional of $\bar{H}$ and $(g, \bar{H}) \geq (\bar{f}, \bar{H})$. On the other hand we have from (6) that $g(x_o^{n_o}) > \bar{f}(x_o^{n_o})$, this contradicts to the maximality of $(\bar{f}, \bar{H})$. Thus we must have $\bar{H} = G$.

Corollary 1 (Sandwich theorem; Kaufman [2], Fuchssteiner [1]). Let f and g be functions of G into $\mathbb{R}$ such that $f \leq g$ and $f(x)+f(y) \leq f(xy)$, $g(x)+g(y) \geq g(xy)$ for all $x, y \in G$. Then there exists a functional h of G such that $f \leq h \leq g$.

Proof. We may assume that G has the identity e and $f(e)=g(e)=0$. The function $\bar{f}$ defined by

$$\overline{f}(x) = \sup\left\{ f(xy)-g(y) \;\middle|\; y \in G \right\} \tag{7}$$

for $x \in G$ is a lower g-function and $f \leq \overline{f}$. Theorem 1 asserts that there is a functional h of G such that $\overline{f} \leq h \leq g$.

We give the following as an application of Theorem 1 to non-negative real valued functions of G. The proof is omitted.

Corollary 2 (Kaufman [3]). Let θ be a function of G satisfying $\theta(xy) \leq \theta(x)+\theta(y)$ for all $x,y \in G$. Let f be a functional of a subsemigroup H of G. Then f is extended to a functional $\overline{f}$ of G such that $0 \leq \overline{f} \leq \theta$ if and only if

$$xh_1=yh_2,\ h_1,h_2 \in H,\ x,y \in G \Longrightarrow f(h_1) \leq \theta(y)+f(h_2). \tag{8}$$

3. Extending non-negative functionals. Let G be a commutative semigroup and H its cofinal subsemigroup (i.e. for any $x \in G$ there is $h \in H$ such that $x|h$ in G). Let f be a non-negative functional of H satisfying

$$h_1|h_2 \text{ in } G \Longrightarrow f(h_1) \leq f(h_2). \tag{9}$$

We define two functions N_f and L_f of G into $\mathbb{R}$ by

$$\begin{aligned} N_f(x) &= \sup\left\{ f(h_1)-f(h_2) \;\middle|\; h_1|xh_2;\ h_1,h_2 \in H \right\}, \\ L_f(x) &= \inf\left\{ f(h_1)-f(h_2) \;\middle|\; xh_2|h_1;\ h_1,h_2 \in H \right\}. \end{aligned} \tag{10}$$

Lemma 2. For all $x,y \in G$ we have

(11) $0 \leq N_f \leq L_f$ and $N_f|_H = L_f|_H = f$.

(12) $N_f(x)+N_f(y) \leq N_f(xy) \leq N_f(x)+L_f(y) \leq L_f(xy) \leq L_f(x)+L_f(y)$.

Inequality (12) implies that N_f is a lower L_f-function and L_f is an upper N_f-function of G. Therefore, we have by Theorem 1

Theorem 2 (Putcha and Tamura [8], Kobayashi and Tamura [7]). Let G be a commutative semigroup and H its cofinal subsemigroup. Let f be a non-negative functional of H. Then f is extended to a non-negative functional of G if and only if f satisfies condition (9).

A cofinal subsemigroup H of G is called strongly cofinal if for every $x \in G$ there are $h \in H$ and $n \in \mathbb{Z}_+$ such that $h \mid x^n$.

Corollary 1. Let f be a positive (positive real valued) functional of a strongly cofinal subsemigroup H of G. Then f is extended to a positive functional of G if and only if f satisfies condition (9).

G is called archimedean if for any $x, y \in G$ there is $n \in \mathbb{Z}_+$ such that $x \mid y^n$. G is called subarchimedean if there is $x_0 \in G$ such that for any $x \in G$, $x \mid x_0^n$ for some $n \in \mathbb{Z}_+$.

Corollary 2. Any positive functional of a subsemigroup of an archimedean commutative semigroup G satisfying condition (9) is extended to a positive functional of G.

4. Existence of non-negative (positive) functionals. Let G be a commutative semigroup. It might not be difficult to describe the condition for G to have non-trivial functionals applying Proposition 1. The problem of finding the concrete condition for G to have non-trivial non-negative (positive) functionals is rather difficult. Tamura and the author [7] gives a necessary and sufficient condition for that in terms of quasi-order. But we do not know the concrete

condition. Some sufficient conditions are obtained from the results in the preceding section.

An element $a \in G$ is called normal if the following two conditions are satisfied;

(13) for any x there is $n \in \mathbb{Z}_+$ such that $x \mid a^n$,

(14) $a^n \mid a^m \Longrightarrow n \leq m$.

These imply that the subsemigroup $[a]$ generated by a is cofinal and that the mapping $a^n \longmapsto n$ is a functional of $[a]$. We can extend the functional to a non-negative functional of G, hence we have

Proposition 2. If a is a normal element of G, then there is a non-negative functional of G such that $f(a) > 0$.

G is called normal (resp. subnormal) if every (resp. some) element of G is normal.

Proposition 3. An archimedean (resp. subarchimedean) commutative semigroup without idempotents is normal (resp. subnormal).

Theorem 3. If G is a normal (resp. subnormal) commutative semigroup, then $\mathrm{Hom}(G, \mathbb{R}_+) \neq \phi$ (resp. $\mathrm{Hom}(G, \mathbb{R}_{+o}) \neq 0$).

In Theorem 3, $\mathbb{R}_+$ (resp. $\mathbb{R}_{+o}$) denotes the additive semigroup of all positive (resp. non-negative) real numbers. The further details of the preceding arguments in §3 and §4 would be found in [7]. By Proposition 3 and Theorem 3 an archimedean commutative semigroup without idempotents is homomorphic into $\mathbb{R}_+$. In particular, an $\mathfrak{N}$-semigroup is homomorphic into $\mathbb{R}_+$, from this we can prove the fundamental fact that an $\mathfrak{N}$-semigroup is a subdirect product of $\mathbb{R}_+$ and an abelian group (Tamura [9], Kobayashi [4]).

In the case of finite rank (the free rank of the quotient group of G is finite) the complete condition for G to have non-trivial homomorphisms into $\mathbb{R}_+$ ($\mathbb{R}_{+o}$) is given as follows.

Theorem 4 (Kobayashi [5]). Let G be a commutative cancellative semigroup of finite rank. Then

(i) $\mathrm{Hom}(G, \mathbb{R}_{+o}) \neq 0$ if and only if G is not a group,

(ii) $\mathrm{Hom}(G, \mathbb{R}_+) \neq \phi$ if and only if G satisfies

(15) for any $x, y \in G$ there exists $n \in \mathbb{Z}_+$ such that $x^{mn} \nmid y^m$ for all $m \in \mathbb{Z}_+$.

The proof is proceeded by reducing to the geometrical consideration in a finite diminsional real vector space. It is necessary for G to be of finite rank because there is a semigroup S of infinite rank satisfying condition (15) and $\mathrm{Hom}(S, \mathbb{R}_{+o}) \neq 0$ (see [5]).

REFERENCES

[1] B. Fuchssteiner, Sandwich theorems and lattice semigroups, J. Functional Analysis 16 (1974), 1-14.

[2] R. Kaufman, Interpolation of additive functionals, Studia Math. 27 (1966), 269-272.

[3] ——————, Maximal semicharacters, Proc. Amer. Math. Soc. 17 (1966), 1314-1316.

[4] Y. Kobayashi, Homomorphisms on N-semigroups into $\mathbb{R}_+$ and the structure of N-semigroups, J. Math. Tokushima Univ. 7 (1973), 1-20.

[5] ——————, Conditions for commutative semigroups to have non-trivial homomorphisms into non-negative (positive) reals, to appear in Proc. Amer. Math. Soc.

[6] ——————, An extension theorem of functionals on commutative semigroups, preprint.

[7] Y. Kobayashi and T. Tamura, Quasi-order preserving homomorphisms of commutative semigroups into the non-negative reals, preprint.

[8] M. S. Putcha and T. Tamura, Homomorphisms of commutative cancellative semigroups into non-negative real numbers, to appear in Trans. Amer. Math. Soc.

[9] T. Tamura, Irreducible $\mathfrak{N}$-semigroups, Math. Nachrichten 63 (1974), 71-88.

Faculty of Education
Tokushima University